AN APPLICATION OF FUNCTIONAL ANALYSIS TO PARTIAL DIFFERENTIAL EQUATIONS IN APPLIED MATHEMATICS[1, 2]

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The three standard second-order partial differential equations with various types of boundary conditions are expressed in languages of Hilbert space theory and the theory of distributions. The concept of strong and weak solutions is explained and then Lions' projection theorem is applied to obtain existence of weak solutions of coercive evolution equations of the parabolic type in the constant domain case. Brief comments are made on recent work on existence and regularity of solutions of variable domain parabolic and hyperbolic coercive equations.

A large number of partial differential equations may be classified into what we call evolution equations or operational differential equations. These are partial differential equations that investigate the time-dependent physical phenomena. We think of the time t as a parameter and examine how the system evolves with time, *i.e.*, we view the system as passing through a succession of states as time increases. Two well-known examples of evolution equations are the wave equation and the heat equation (or the diffusion equation).

Let $\underset{\sim}{x}$ denote the space variable x or (x_1,x_2) or (x_1,x_2,x_3) as the case may be.

The wave equation (which is a hyperbolic equation) is,

$$\frac{\partial^2 u(\underset{\sim}{x},t)}{\partial t^2} - c^2 \Delta u(\underset{\sim}{x},t) = f(\underset{\sim}{x},t), \tag{1}$$

$\forall\, \underset{\sim}{x}$ in a certain region D of space and $\forall\, t >$ a certain time T. We may take $T=0$. In a general situation, the speed c of propagation of waves may not remain constant as $\underset{\sim}{x}$ or t varies. We are confronted with the wave equation (1) when, for example, we want to determine the small transverse deflection $u(\underset{\sim}{x},t)$ of a stretched membrane (or a string) subject to a forcing function $f(\underset{\sim}{x},t)$, under the initial conditions,

$$\left.\begin{aligned} u(\underset{\sim}{x},0) &= g_1(\underset{\sim}{x}) \\ \frac{\partial u}{\partial t}(\underset{\sim}{x},0) &= g_2(\underset{\sim}{x}) \end{aligned}\right\} \quad \forall\, \underset{\sim}{x} \in D,$$

and the boundary condition,

$$u(\underset{\sim}{x},t) = h(\underset{\sim}{x},t) \quad \forall\, t>0 \quad \forall\, \underset{\sim}{x} \in \partial D, \text{ the boundary of D.}$$

Here g_1, g_2, h are given functions.

The equation of heat conduction (which is a parabolic equation) states that the temperature $u(\underset{\sim}{x},t)$ of the medium of heat conduction satisfies the partial differential equation,

$$\frac{\partial u(\underset{\sim}{x},t)}{\partial t} - \alpha\, \Delta\, u(\underset{\sim}{x},t) = f(\underset{\sim}{x},t), \tag{2}$$

$\forall\, t>T$ and $\forall\, \underset{\sim}{x} \in D =$ the region of space defining the extent of the medium of heat conduction. Here also, in a general situation α may be a nonconstant function of $\underset{\sim}{x}$

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and t. ($\alpha = \frac{k}{c\rho}$ where k is the thermal conductivity, c is the capacity and ρ is the density of the substance through which heat is flowing). The source function f gives the amount of heat produced within the conducting medium per unit length per unit time.

Consider the following particular situation of heat conduction:

The conducting medium occupies the region D of the xy-plane. The initial condition is $u(\underset{\sim}{x},T) = g(\underset{\sim}{x})\ \forall\ \underset{\sim}{x} \in D$, and the boundary condition is

$$u(\underset{\sim}{x},t) = h(\underset{\sim}{x},t) \quad \forall\ \underset{\sim}{x} \in \partial D \text{ and } \forall\ t \in [T,T_1]. \tag{3}$$

Here g and h are given. The region in space-time for which we want to solve the heat equation is, therefore, a cylinder of the type shown, with axis parallel to the t-axis (figure 1).

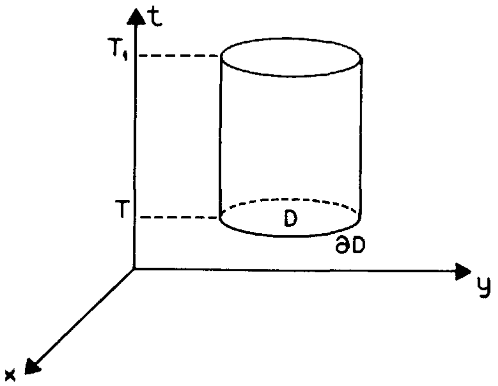

FIGURE 1. Illustration of the space-time region for which the heat equation is to be solved.

The situation is altered if we replace the boundary condition (3) for some or all of $(\underset{\sim}{x},t) \in \partial D \times [T,T_1]$, by say $\frac{\partial u}{\partial n}(\underset{\sim}{x},t) = h_1(\underset{\sim}{x},t)$. Thus the nature of the boundary conditions may actually change from time to time, giving rise to what is known as a variable domain situation. In a constant domain situation, the boundary condition remains constant for all time. If the boundary condition varies too wildly with time, the heat equation may not have a solution at all. In as much as $u(\underset{\sim}{x},t)$ is a continuous function with possibly continuous partial derivatives, we have to impose certain smoothness on the variation of the boundary condition with time, in order to make sure that solution exists.

Now let me give examples of partial differential equations which are not of the evolution type, namely Laplace's equation or Poisson's equation $\Delta u(\underset{\sim}{x}) = f(\underset{\sim}{x})$. These differential equations describe time-independent physical situations, and appear in the potential theory or in the theory of steady-state heat conduction.

These are examples of elliptic differential equations. The examples given above were all linear. Here are some examples of nonlinear differential equations:

A. Navier-Stokes's equations arising in incompressible fluid dynamics. In a two-dimensional flow, the fluid velocity vector $\underline{u} = iu_1 + ju_2$ satisfies the equation,

$$\frac{\partial \underline{u}}{\partial t} - \rho\left(\frac{\partial^2 \underline{u}}{\partial x_1^2} + \frac{\partial^2 \underline{u}}{\partial x_2^2}\right) - \left(u_1 \frac{\partial \underline{u}}{\partial x_1} + u_2 \frac{\partial \underline{u}}{\partial x_2}\right) = -\text{grad } p + \underline{f},$$

where $\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0$. Here ρ is the viscosity, p is the pressure and $f(x_1,x_2,t)$ is some external force. One can consider suitable initial and boundary conditions, *e.g.*, $\underline{u}(x_1,x_2,0) = \underline{u}_0$ on Ω and $\underline{u}(x_1,x_2,t) = 0$ on $\partial\Omega \times [0,T]$.

B. The following equation has appeared in the nonlinear meson theory of nuclear forces,

$$\frac{\partial^2 u}{\partial t^2} - \Delta\, u + m^2 u + k|u|^2 u = f.$$

In all the examples above, the functions f, f_1, f_2, g, g_1, g_2, h, h_1, etc., constitute the data. Usually the data satisfy certain properties of integrability, continuity or differentiability. The solution u that we look for becomes useful in concrete physical situations if it also has some such properties, *e.g.*, we may want u to be integrable, square-integrable, or bounded in $\underset{\sim}{x}$, the space variable, or we may want u to have space derivatives which are integrable, square-intergrable or bounded in space variable. In evolution equations, u will, of course, be differentiable with respect to time, the derivative in turn satisfying some or all of the conditions mentioned above.

One way to adapt the Hilbert space techniques of functional analysis is to obtain the existence of solutions to the linear equations mentioned above. In a few cases, this theory actually succeeds in describing a process of constructing the solution. With this end in view, we first rewrite the differential equations in the language of Hilbert space theory. As an example, a heat conduction problem may be presented as follows:

Let Ω be a bounded open subset of R^3. Let $H=L^2(\Omega;C)$, $H^1=\{u \in H \mid u_{x_1}, u_{x_2}, u_{x_3} \in H\}$ and $H_0^1=\{u \in H \mid \Omega\ u$ is zero on $\partial\Omega\}$. It is known that H is a Hilbert space under the inner product

$$(u,v)=\int_\Omega u(x_1,x_2,x_3)\,\overline{v(x_1,x_2,x_3)}\,d\underset{\sim}{x},$$

and H^1, H_0^1 are Hilbert spaces under the inner product

$$((u,v)) = (u,v)+(u_{x_1},v_{x_1})+(u_{x_2},v_{x_2})+(u_{x_3},v_{x_3}).$$

H^1 and H_0^1 are particular cases of Sobolev spaces.

We can now state a heat conduction problem to be:

"To show the existence of a

$$u \in L^2([T,T_1];\ H_0^1)$$

with

$$u_t, u_{x_1x_1}, u_{x_2x_2}, u_{x_3x_3} \in L^2([T,T_1];\ H) \tag{4}$$

so that

$$u_t-\alpha\,\Delta\,u=f, \tag{5}$$

and

$$u(\underset{\sim}{x},T)=u_0 \text{ (an initial condition)}$$

where f is a given function in $L^2([T,T_1];\ H)$. Note that the very fact that $u \in L^2([T,T_1];\ H_0^1)$ implies that the boundary condition

$$u(\underset{\sim}{x},t)\Big|_{\underset{\sim}{x}\in\partial\Omega}=0 \quad \forall\, t \in [T,T_1]$$

is satisfied. For simplicity, let us suppose that α is a constant >0." It suffices to take all the derivatives in the sense of distribution in appropriate variables.

Our object, obviously, is to exploit to our advantage the properties of the Hilbert spaces H^1 and H_0^1—namely, the properties of inner product and orthogonal projection. In this way, we will also tackle the difficult conditions (4). Let us now see how it can be done.

If $v \in L^2([T,T_1]:\ H_0^1)$ with $v_t \in L^2([T,T_1];\ H)$, then we have from (5),

$$\int_T^{T_1}(u_t(t),v(t))_H dt-\alpha\int_T^{T_1}(\Delta u(t),v(t))_H dt=\int_T^{T_1}(f(t),v(t))_H dt \tag{6}$$

By integration by parts we obtain,

$$\int_T^{T_1} (u_t(t),v(t))_H dt = (u(T_1),v(T_1))_H - (u(T),v(T))_H - \int_T^{T_1} (u(t),v_t(t))_H dt$$

and

$$(\Delta u(t),u(t))_H = \sum_{i=1}^{3} \int_\Omega \frac{\partial^2 u(\underset{\sim}{x},t)}{\partial x_i^2} \overline{v(\underset{\sim}{x},t)}\, d\underset{\sim}{x} = -\sum_{i=1}^{3} \int_\Omega \frac{\partial u(\underset{\sim}{x},t)}{\partial x_i} \frac{\overline{\partial v(\underset{\sim}{x},t)}}{\partial x_i}\, d\underset{\sim}{x}$$

because $v(\underset{\sim}{x},t)$ vanishes on $\partial\Omega$. Thus, if u is a solution of equation (5), then u satisfies equation (6) which reduces to

$$\text{(7)} \quad \begin{cases} -\int_T^{T_1} (u(t),v_t(t))_H dt + \alpha \int_T^{T_1} [\sum_{i=1}^{3} (u_{x_i},v_{x_i})]\, dt \\ = \int_T^{T_1} (f(t),v(t))dt + (u_o,v(T))_H \\ \forall\, v \in L^2([T,T_1];\, H_0^1) \text{ satisfying} \\ v_t \in L^2([T,T_1];\, H) \text{ and } v(T_1)=0. \end{cases}$$

It helps us now to define the weak problem corresponding to the strong problem (5)

Weak problem. To find a $u \in L^2([T,T_1];\, H_0^1)$ such that the equation (7) is satisfied.

It is clear that the existence of a solution to the strong problem implies the existence of a solution of the corresponding weak problem. The converse is, in general, false, precisely because we lack the information of whether the solution u of the weak problem satisfies the additional condition $u_t \in L^2([T,T_1];\, H)$.

Now we split the problem of solving an original strong problem into two parts:

A. Whether the corresponding weak problem has a solution, and if the answer is yes,

B. Whether this weak solution is actually a strong solution.

In the special heat conduction problem we considered above, a direct application of Lions' much-used projection theorem provides an affirmative answer to part A. Part B needs some more work to answer it. It has very recently been shown that if $f' \in L^2([T,T_1];\, H)$ with $f(T)=0$, then the answer to part B is also "yes".

One may wonder why such a devious method is adopted to solve a simple equation such as (5), $u_t - \alpha\,\Delta\,u = f$, α being a constant. The reason is that we want to handle much more general situations, when α is not a constant or when operators other than $-\Delta$ are involved. Our aim is to give a general approach which is common to a large number of situations, instead of devising a separate trick to handle even a slightly different situation. In fact, letting H be a separable Hilbert space, Lions' projection theorem gives the existence of a solution $u \in L^2([T,T_1];\, H_0^1)$ of any weak equation of the type,

$$\text{(8)} \quad \begin{cases} -\int_T^{T_1} (u(t),v_t(t))dt + \int_T^{T_1} b(t;u(t),v(t))dt \\ \qquad = \int_T^{T_1} (f(t),v(t))dt + (u_o,v(T)) \\ \forall\, v \in L^2([T,T_1];\, H_0^1) \text{ satisfying } v_t \in L^2([T,T_1];\, H) \\ \quad \text{with } v(T_1)=0, \end{cases}$$

whenever, $\forall\, t \in [T,T_1]$, $b(t;\cdot,\cdot) : H_0^1 \times H_0^1 \to C$ is a continuous sesquilinear form satisfying the condition, called the coercivity condition,

$$\text{Re}\, b(t;x,x) \geq \alpha \|x\|^2 \quad \forall\, x \in H_0^1 \tag{9}$$

for some constant $\alpha > 0$.

In fact we could replace H_0^1 by any other Hilbert space V such that V is a dense subspace of H with continuous inclusion injection: $V \to H$.

Now we state Lions' projections theorem:

Theorem. Let H be a Hilbert space and V a pre-Hilbert space such that V is a subspace of H with continuous inclusion injection: $V \to H$. Let the norm in V be denoted by $|||\cdot|||$. Let E: $H \times V \to C$ be a sesquilinear form continuous in the first variable such that,

$$|E(u,v)| \geq \alpha |||v|||^2 \quad \forall\, v \in V$$

for some constant $\alpha > 0$. Then for every continuous anti-linear form L: $V \to C$, there exists a $u_L \in H$ such that,

$$E(u_L,v) = L(v) \quad \forall\, v \in V.$$

This completes the part of the statement of the theorem that we need.

Lions' projection theorem is a generalization of the well-known Lax-Milgram theorem of Hilbert spaces and was itself generalized in 1970 and 1972 to cover certain noncoercive situations as well.

Using Lions' projection theorem, we will now prove (8) when (9) holds.

Let $H = L^2(T,T_1]; V)$ where V is H_0^1 or H^1 or any other Hilbert space which is a dense subspace of H with continuous inclusion injection. Let

$$V = \{u \in H \mid u_t \in L^2([T,T_1]; H),\ u(T_1) = 0\}.$$

H is a Hilbert space under the inner product

$$((u,v))_H = \int_T^{T_1} (\,(u(t),v(t)\,)\,)_V\, dt,$$

and V is a Hilbert space under the inner product

$$(((u,v)))_V = ((u,v))_H + (u(T),v(T)\,)_H.$$

Clearly V is a subspace of H with continuous inclusion injection: $V \to H$. Let

$$E: H \times V \to C$$

be defined by

$$E(u,v) = -\int_T^{T_1} (u(t),v_t(t)\,)_H dt + \int_T^{T_1} b(t;u(t),v(t)\,)\, dt. \tag{10}$$

E is obviously a sesquilinear form continuous in the first variable. Therefore, $\forall\, v \in V$,

$$\begin{aligned} 2\,\text{Re}\, E(v,v) &= -\int_T^{T_1} \frac{d}{dt}(v(t),v(t)\,)dt + 2\,\text{Re} \int_T^{T_1} b(t;v(t),v(t)\,)dt \\ &= |v(T)|_H^2 + 2\,\text{Re} \int_T^{T_1} b(t;v(t),v(t)\,)dt \\ &\geq |v(T)|_H^2 + 2\alpha \int_T^{T_1} \|v(t)\|_V^2\, dt \quad \text{by (9).} \end{aligned}$$

$$\therefore \operatorname{Re} E(v,v) \geq \frac{1}{2}|v(T)|_H^2 + \alpha||v||_H^2$$

$$\geq c(|v(T)|_H^2 + ||v||_H^2)$$

where $0 < c = \min\left(\frac{1}{2}, \alpha\right)$.

Thus $\operatorname{Re} E(v,v) \geq c|||v|||_V^2$, whence

$$|E(v,v)| \geq c|||v|||_V^2 \quad \forall\, u \in V.$$

Defining $L\colon V \to C$ by

$$L(v) = \int_T^{T_1} (f(t), v(t))_H dt + (u_0, v(T)), \tag{11}$$

it is easily seen that L is anti-linear, and

$$|L(v)| \leq \left(\int_T^{T_1} |f(t)|_H^2 dt\right)^{1/2} \left(\int_T^{T_1} |v(t)|_H^2 dt\right)^{1/2} + |u_0|_H |v(T)|_H$$

$$\leq k\left[\left(\int_T^{T_1} ||v(t)||_V^2 dt\right)^{1/2} + |v(T)|_H\right]$$

$$\leq \sqrt{2}\, k|||v|||_V$$

so that L is continuous also. Hence, by Lions' projection theorem, there exists a $u \in H$ such that

$$E(u,v) = L(v) \quad \forall\, v \in V,$$

which when written out fully, using (10) and (11), yields (8). This completes the proof of the existence of a solution of the weak coercive parabolic problem in constant domain situation.

Several other problems can be handled in this manner—some of them easy and some difficult. In the "easy" classification falls the weak coercive hyperbolic problem with constant domain. All one has to do is to define suitable spaces H, V, and then construct suitable maps E and L as a starting point. All noncoercive problems may be classified as difficult, but since 1970, ways have been discovered to apply Lions' projection theorem to noncoercive problems.

All the problems mentioned above have their variable domain counterparts. For example, we could require the solution u to satisfy $u(t) \in H_0^1$ for some t's and $u(t) \in H^1$ for other t's. In a general situation, we replace the single Hilbert space V by a family $\{V(t) | t \in [T,T_1]\}$ of Hilbert spaces, each $V(t)$ being a dense subspace of H with continuous inclusion injection. Until now, probably the best way to attack such problems is to make use of the family $\{S(t) | t \in [T,T_1]\}$ of certain uniquely defined unbounded linear operators in H, the action of $S(t)$ being described by $(S(t)x, S(t)y)_H = ((x,y))_{V(t)}$ $\forall x,y \in$ domain of $S(t)$. The existence of such operators is a standard proposition in Hilbert space theory. Carroll pointed out that the domain of $S(t)$ is exactly $V(t)$, and he was the first to show how these operators could be exploited in the problems we are interested here. Note that $S(t)^{-1}:H\rightarrow V(t)$ is a continuous surjection. Solutions to our variable domain evolution problems exist if $V(t)$ varies in certain smooth ways, and this smoothness is expressed by placing appropriate weak continuity or weak differentiability conditions on the family $\}S(t)^{-1}|t \quad [T,T_1]\}$. The existence problem in the variable domain parabolic coercive case becomes very simple and is handled in exactly the same way as we earlier handled the constant domain parabolic coercive case. The variable domain hyperbolic coercive problem is more complicated, and the existence question in this case was settled (at least in part) by Carroll and State in 1971. Questions other than that of existence, like regularity questions are being handled now and some nontrivial results have been obtained very recently. Noncoercive situations promise to be much more difficult. We will not go deeper into these questions here.

Let us now turn to a simpler situation. Instead of considering the problem

$$u_t-\alpha\,\Delta\,u=f,$$

let us consider the elliptic problem

$$-\Delta\,u=f, \tag{12}$$

where $f\in H=L^2(\Omega;C)$, Ω being a bounded open set in R^3 with a nice boundary. In what follows we will make use of Green's formula, true for all $v\in H^1$, namely that

$$-\int_\Omega\left(\frac{\partial^2u}{\partial x_1^2}+\frac{\partial^2u}{\partial x_2^2}+\frac{\partial^2u}{\partial x_3^2}\right)\bar v\,d\underset{\sim}{x} \tag{13}$$

$$=\int_\Omega\left(\frac{\partial u}{\partial x_1}\overline{\frac{\partial v}{\partial x_1}}+\frac{\partial u}{\partial x_2}\overline{\frac{\partial v}{\partial x_2}}+\frac{\partial u}{\partial x_3}\overline{\frac{\partial v}{\partial x_3}}\right)d\underset{\sim}{x}+\int_{\partial\Omega}\frac{\partial u}{\partial n}\bar v\,d\sigma$$

where $\hat n$ is the unit interior normal and $\int d\sigma$ denotes integration over the boundary $\partial\Omega$ of Ω.

We will now state two kinds of weak problems that correspond to the strong problem (12); many other kinds exist:

1) Problem of existence of $u\in H_0^1$ such that

$$\int_\Omega\left(\frac{\partial u}{\partial x_1}\overline{\frac{\partial v}{\partial x_1}}+\frac{\partial u}{\partial x_2}\overline{\frac{\partial v}{\partial x_2}}+\frac{\partial u}{\partial x_3}\overline{\frac{\partial v}{\partial x_3}}\right)d\underset{\sim}{x}=\int_\Omega f\,\bar v\,d\underset{\sim}{x}\quad\forall\, v\in H_0^1;\text{ and,}$$

2) Problem of existence of $u\in H^1$ such that

$$\int_\Omega\left(\frac{\partial u}{\partial x_1}\overline{\frac{\partial v}{\partial x_1}}+\frac{\partial u}{\partial x_1}\overline{\frac{\partial v}{\partial x_2}}+\frac{\partial u}{\partial x_3}\overline{\frac{\partial v}{\partial x_3}}\right)d\underset{\sim}{x}=\int_\Omega f\,\bar v\,d\underset{\sim}{x}\quad\forall\, v\in H^1. \tag{14}$$

Let us note that in the first problem, the solution u, if it exists, is an element of H_0^1, which implies that u is zero on the boundary $\partial\Omega$ of Ω. The first problem is therefore the Dirichlet problem.

To show the existence of solutions of these two problems, one could utilize Lions' projection theorem. But more fundamental results in the Hilbert space theory and the distribution theory allow us to deduce that not only a solution u to each problem exists, but that actually this is a strong solution, *i.e.*, actually we have

$$-\Delta\,u=f,$$

derivatives being taken in the distribution sense. This, together with (13), help us to rewrite (14) as

$$-\int_{\Omega} \Delta u \, \bar{v} \, d\underset{\sim}{x} - \int_{\partial\Omega} \frac{\partial u}{\partial n} \bar{v} \, d\sigma = -\int_{\Omega} \Delta u \, \bar{v} \, d\underset{\sim}{x} \quad \forall \, v \in H^1.$$

Hence, $\int_{\partial\Omega} \frac{\partial u}{\partial n} \bar{v} \, d\sigma = 0 \quad \forall \, v \in H^1$. This, therefore, implies that in a certain sense $\frac{\partial u}{\partial n} = 0$ on the boundary $\partial\Omega$ of Ω, and we have solved the Neumann problem.

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